Lecture 30 - Simple harmonic motion(SHM)

Simple harmonic is the simplest model possible of oscillatory motion, yet it is extremely important. Examples include the motion of the pendulum in a grandfather clock, the vibrations of atom inside a crystal, the oscillations of a buoy due to wave motion in a lake and as we shall discover it forms the basis for understanding wave motion itself. In fact, each point on a wave undergoes simple harmonic motion.

To understand simple harmonic motion, we will study in some detail a mass attached to a spring. The ideas we develop for this system will be applied to all other cases which we consider, so it is very important to understand this system very well. To get started we recall the restoring force which a spring exerts when it is displaced from its equilibrium position by a distance x. The force which the spring produces is given by,

$$F_s = -kx \tag{1}$$

k is the spring constant. The larger k is the "stiffer" the spring and the stronger the restoring force. In the study of mechanics, we also figured out the energy stored in a spring and found that it was given by,

$$PE = \frac{1}{2}kx^2. \tag{2}$$

We found this result by finding the area under the graph of F_s versus x. Consider attaching a mass to the end of the spring and suspending the spring from a support. The equilibrium extension of the spring is given by, $mg - kx_{equil} = 0$, so that

$$x_{equil} = \frac{mg}{k} \tag{3}$$

Simple harmonic motion describes motion about the equilibrium position. It is best to describe this motion by first changing the origin so that $x_{equil} = 0$. Now we may displace the spring from equilibrium by a distance x and Eqs. (1) and (2) apply to the deviation from the equilibrium position. If we extend the spring and then release it, the resulting oscillatory motion is the simplest example of simple harmonic motion(SHM). The vibrations of atoms in a crystal about their equilibrium positions is very similar to this spring motion. Note that in our example, it is traditional to use x even though our spring is oscillating in the vertical direction. There are many different ways to initiate the SHM, for example we could give the mass an initial velocity instead of displacing it. The way in which the SHM is started is called the initial condition.

As SHM procedes, there is a continuous transfer of energy from potential energy stored in the spring to kinetic energy of motion. To understand this energy transfer, note that during SHM the mass has special times when it is at its equilbrium position. At these times, $x = 0$, so all of its energy is kinetic energy. At other times, the mass is at its maximum distance from the equilibrium point and this maximum distance is called the amplitude A of the SHM. If the maximum velocity is v_0, energy conservation requires that,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kA^2 \tag{4}$$

Since the motion changes with time, the position and velocity of the mass are time dependent so we consider the variables $x(t)$, $v(t)$ and $a(t)$ to describe the kinematics of the motion. However, note that the acceleration $a(t)$ is NOT constant. At any time however energy is conserved so that

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 \tag{5}$$

Solving for v we then find that,

$$v = (\frac{k}{m})^{1/2}(A^2 - x^2)^{1/2} \tag{6}$$

From the analogy with circular motion, we have,

$$v_0 = \frac{2\pi A}{T} = 2\pi f A = \omega A \tag{7}$$

Combining Eq. (4) and (7) we then find that the angular frequency of SHM in a mass-spring system is given by,

$$\omega = (\frac{k}{m})^{1/2}; \quad \text{so that} \quad f = \frac{1}{2\pi}(\frac{k}{m})^{1/2} \quad \text{and} \quad T = 2\pi(\frac{m}{k})^{1/2} \tag{8}$$

where T is the period, f is the frequency and has units of Hertz (Hz), and ω is the angular frequency (rad/s). From the analogy with circular motion, we find that,

$$x(t) = ACos(\omega t + \theta_0) \tag{9}$$

Using this equation in Eq. (6) and trig identities, it is possible to show that,

$$v(t) = -A\omega Sin(\omega t + \theta_0) = -v_0 Sin(\omega t + \theta_0) \tag{10}$$

and finally by using $a = -kx/m$, it is seen that the acceleration is given by,

$$a(t) = -A\omega^2 Cos(\omega t + \theta_0) = -a_0 Cos(\omega t + \theta_0) \tag{11}$$

These equations contain the complete solution to SHM are are used in many different situations.